Selective Lamb mode excitation by piezo-electric coaxial ring actuators

E.V. Glushkov[1], **N.V. Glushkova**[1], **O.V. Kvasha**[1], **and R. Lammering**[2]

[1] Kuban State University, P.O. Box 1581, Krasnodar 350080, Russia
e-mail: evg@math.kubsu.ru

[2] Institute of Mechanics, Helmut-Schmidt-University/University of the Federal Armed Forces, D-22043 Hamburg, Germany

Abstract. The article describes an omni-directional multi-element transducer for selective Lamb wave mode excitation. It is composed of several coaxial ring-shaped piezoelectric elements actuated by n-cycled sinusoidal tonebursts. Mode selection is achieved by a special choice of the time delays and the amplitudes of the input driving signals. The method for the determination of these parameters is based on strict analytical considerations. In the limiting case of the infinite number of cycles and with a sufficient number of actuators it allows to generate only a single required mode, and to completely eliminate undesirable ones. It is shown that within the range of applicability of the simplest model for the piezoelectric elements, i.e. when the actuating force can be replaced by the concentrated forces applied at the locations of the edges of an element, no time delays are needed, and the required mode of excitation is achieved only by an approprite choice of the amplitudes of the input signals.

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1. Introduction

An important part in the design of a guided wave based active structural health monitoring system is the development of an effective excitation scheme: the transducer should be capable of generating waves of detectable amplitudes and, at the same time, be compact and lightweight enough, so that it could be permanetly attached to the structure. Comprehensive reviews on various transducer types, as well as on the state of art in guided wave structural health monitoring as a whole, are [1, 2, 3]. One of the most popular solutions has been the use of piezoelectric lead zirconate titanate (PZT) elements. The more modern alternatives include polyvinylidene fluoride (PVDF) polymer film and various types of piezocomposite transducers, such as active and macro fiber composite ones (AFCs and MFCs). All these devices provide excellent performance in guided wave excitation and acquisition together with neglectable mass and volume as well as low cost.

However, guided waves generated by a piezoelectric patch actuator in a waveguide unavoidably contain multiple modes. Therefore, special efforts for transducer design have to be made in order to achieve the excitation of the single desirable mode at a required frequency band and avoid sophisticated signal processing procedures. One of the possible approaches here is to use a single element with a special selection of its size. As is well known, if the distance between the edges of the patch is approximately equal to a half-integer number of the wavelength of a Lamb mode propagating in an elastic plate, it provides the maximum to the energy carried by this wave mode in the two-mode frequency range (e.g. [4, 5]). Similarly, with the size of the actuator equal to an integer number of the wavelengths, minimal values of energy are achieved. Therefore, it is possible to select the size of a patch in such a way that only the one required mode will be generated at the given frequency. These results have received further experimental confirmation in [6, 7].

The more flexible approach is to compose a transducer from multiple separately driven elements. Thus, in the article [8] a pair of piezoelectric patch actuators actuated in-phase by Hann-windowed sinusoidal tonebursts was used in order to implement Lamb mode selection in a composite laminated plate. It was found that certain distances between the elements gave a maximum normal displacemnt of a_0 mode and a minimal displacement of the s_0, providing a way to produce pure a_0 mode at a given central frequency of the input signals. At that, theoretical results were confirmed by experimental investigations. In the article [9] the same goals were achieved by a pair of elements symmetrically mounted on the top and the bottom surfaces of a plate. Being actuated in-phase or out-of-phase, they produce pure symmetrical or antisymmetrical motions, respectively, allowing to excite a single Lamb wave mode using an excitation signal with the frequency spectra within the two-mode frequency range.

Within the indicated approaches mode selection is achieved only by a proper arrangement of piezoelectric elements being excited simultaneously by identical in-phase or out-of-phase driving signals. The problem can be considered from another point of view. It is natural to assume a predetermined geometrical configuration of the system and search for appropriate driving input signals in order to obtain the desirable mode of excitation. For example, in [10] guided wave mode selection in a carbon steel pipe was implemented by appling appropriately selected time-delayed input tone-burst signals to a phased comb transducer array. In this brief note such an approach is applied for the development of an omni-directional transducer for excitation of selected Lamb wave modes in an elastic plate. It is composed of several coaxial ring-shaped piezoelectric actuators excited by n-cycled sinusoidal tonebursts with an identical central frequency. Generation of a required mode is achieved using the driving input signals with appropriately selected time delays and amplitudes.

The method of selection of the parameters of the driving signals given in this paper is based on strict analytical considerations. It comes from the observation that with a large enough number of cycles the initially non-stationary problem is closely related to a certain time-harmonic problem, and so the sought-for amplitudes and time delays can be approximately expressed through the amplitudes and phase shifts obtained from this latter one. For the case of piezoelectric strip actuators similar time harmonic problems were intensively investigated in our previous articles [5, 11]. In particular, it was established that if the number of the actuators is greater then the number of the undesirable Lamb wave modes, it is always possible to eliminate them completely by a special choice of the amplitudes and phase shifts of the driving voltages. Similar results are obtained here for the coaxial system of ring-shaped piezoelectric actuators. In the non-stationary case this of course holds only approximately. It is also unexpectedly found that within the assumptions of the used model no time delays are needed at all, it est the selective mode excitation is achieved only by an appropriate choice of the amplitudes of the driving signals.

2. Statement of the problem

Let us consider a homogeneous isotropic elastic plate ($0 < r < \infty$, $0 \leq \varphi \leq 2\pi$, $-h \leq z \leq 0$) with a system of thin and flexible ring-shaped coaxial piezoelectric actuators bonded to its top surface $z = 0$ at the areas Ω_m: $r_{1m} \leq r \leq r_{2m}$, $m = 1, ..., M$. For determinacy, let the actuators be fabricated from piezoceramic material of class 6mm poled in z direction. The modeling of piezoelectric films of other kinds should not require considerable modifications of the method developed.

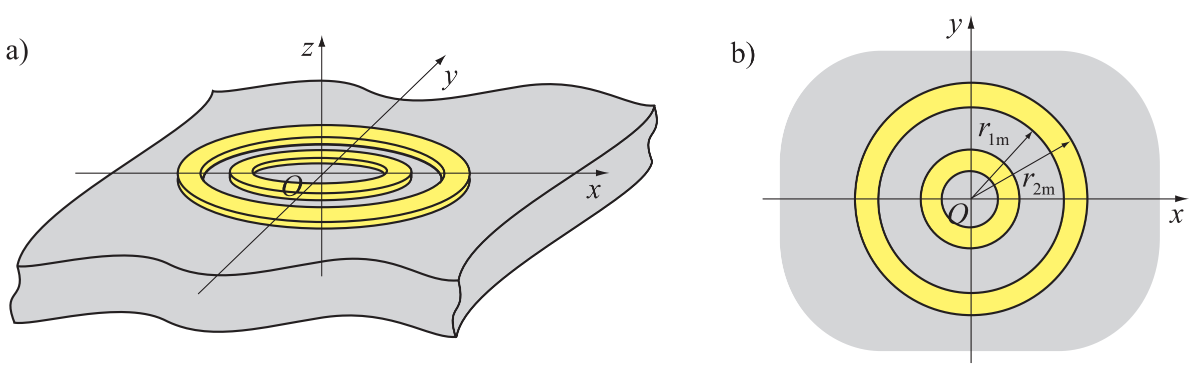

Figure 1. Elastic plate with piezoelectric actuators.

With driving voltages $V_m(t)$, $m = 1, ..., M$ applied to the electrodes, the actuators expand and contract, generating an axially-symmetric displacement field $\mathbf{v}(r, z, t)$ in the plate. Within the assumptions of the linear theory of elasticity it obeys the Lamé equations

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{v} + \mu\Delta\mathbf{v} = \rho\frac{\partial^2\mathbf{v}}{\partial t^2} \tag{2.1}$$

and the boundary conditions

$$\boldsymbol{\sigma}|_{z=0} = \mathbf{q}(r, t), \qquad \boldsymbol{\sigma}|_{z=-h} = 0, \tag{2.2}$$

where $\mathbf{q}(r,t)$ is the traction generated by the actuators; λ, μ are the Lamé constants of the plate, ρ is its density, $\boldsymbol{\sigma} = (\tau_{rz}, \sigma_z)^T$ is the stress vector at a horizontal surface element.

We suppose that the traction is represented as a linear combination

$$\mathbf{q}(r,t) = \sum_{m=1}^{M} V_m(t)\mathbf{q}_m(r), \tag{2.3}$$

where $\mathbf{q}_m(r,t)$ is the contact stress generated by the whole system of the actuators with the unity voltage applied to the m-th of actuator. This is a very general assumption, and it holds in the context of the linear theory of piezoelectricity.

Nevertheless, for the sake of simplicity we shall use a simplified engineering model for the actuators. Within its bounds, the action of an actuator is modeled by the tangential traction of uniform magnitude applied along its edges. As a result

$$V_m(t)\mathbf{q}_m(r) = V_m(t)(q_{rm}(r), 0)^T, \tag{2.4}$$

where

$$q_{rm}(r) = \tau_0(-\delta(r - r_{1m}) + \delta(r - r_{2m})). \tag{2.5}$$

Therefore, the dynamics of the actuators and their mutual interacion are neglected, and it is assumed that the plate dynamics is uncoupled from the actuators dynamics. This approximation is enough for many practical applications, and works satisfactory in the two-mode frequency range. With more details its applicability range is discussed in [5, 12].

Outside the actuation area, the solution to the problem (2.1)-(2.2) is represented as a superposition of the normal modes of the plate:

$$\mathbf{v}(r,z,t) = \sum_{k=1}^{\infty} \mathbf{v}_k(r,z,t), \quad r \geq r_{2M}. \tag{2.6}$$

Our main goal is to obtain the vector of functions $\mathbf{V}(t) = (V_1(t), \ldots, V_M(t))$ providing the selective radiation of the modes with a given numbers $k \in H$. Namely, we need to maximize the functional

$$\nu(\mathbf{V}(t)) = \frac{\sum\limits_{k\in H} E_k(\mathbf{V}(t))}{\sum\limits_{k\in N\setminus H} E_k(\mathbf{V}(t))} \tag{2.7}$$

representing the contrast of the radiation during the actuation process, where E_k is the total amount of energy carried out to infinity by the k-th mode.

To complete the statement of the optimization problem, we need to specify a functional space for the driving voltages. We assume that all the input signals have the same form determined by a smooth function $f_n(t)$, with possible variations in amplitudes A_m and time delays t_m:

$$V_m(t) = A_m f_n(t - t_m), \quad -\infty < A_m < \infty, \quad 0 \leq t_m < \infty, \tag{2.8}$$

and the frequency spectras

$$\hat{f}_n(\omega) = \int_0^\infty f(t)e^{i\omega t}dt \tag{2.9}$$

of the functions $f_n(t)$ form a delta like sequence, i.e. with the increase of n they become more and more concentrated near the point $\omega = \omega_c$. These assumptions are practical and quite general. For example, widely used n-cycle sinusoidal tonebursts

$$f_n(t) = \begin{cases} \sin 2\pi t/T, & 0 \le t \le nT \\ 0, & t < 0 \text{ or } t \ge nT \end{cases} \tag{2.10}$$

with a given central circular frequency $\omega_c = 2\pi/T$ have the frequency spectras

$$\hat{f}_n(\omega) = -\frac{\omega_c}{\omega^2 - \omega_c^2}(1 - e^{i\omega nT}), \tag{2.11}$$

and clearly hold the specified property.

3. Structure of the wavefields

The non-stationary problem (2.1)-(2.2) can be reduced to the time-harmonic one using the Fourier transform with respect to the time variable t:

$$\mathbf{u}(r,z,\omega) = \int_0^\infty \mathbf{v}(r,z,t)e^{i\omega t}dt. \tag{3.1}$$

This leads to the representation of any arbitrary transient motion in terms of steady-state harmonic oscillations $\mathbf{u}(r,z,\omega)e^{-i\omega t}$:

$$\mathbf{v}(r,z,t) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \mathbf{u}(r,z,\omega)e^{-i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{u}(r,z,\omega)e^{-i\omega t}d\omega. \tag{3.2}$$

Similar representations take place for other field variables. In particular, the stress vector $\boldsymbol{\sigma}_n(r,z,t)$ at an area element with a normal n can be expressed via its complex amplitude $\boldsymbol{\tau}_n(r,z,\omega)$. The same is for the voltages $V_m(t)$ and their frequency spectras $\hat{V}_m(\omega)$.

General solution to the resulting time-harmonic problem is well known. In the present case it has the form

$$\mathbf{u}(r,z) = \frac{1}{2\pi}\int_\Gamma \mathbf{i}^{-1}\mathbf{J}(\alpha r)K(\alpha,z)\sum_{m=1}^M \hat{V}_m\mathbf{Q}_m(\alpha)\,\alpha d\alpha, \tag{3.3}$$

$$\mathbf{Q}_m(\alpha) = 2\pi\int_0^\infty \mathbf{i}\mathbf{J}(\alpha r)\mathbf{q}_m(r)rdr, \tag{3.4}$$

where

$$\mathbf{J}(\alpha r) = \begin{pmatrix} J_1(\alpha r) & 0 \\ 0 & J_0(\alpha r) \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix};$$

the matrix $K(\alpha,z)$ has the view

$$K = \begin{pmatrix} -iM_n & -i\alpha P \\ S_n/\alpha & R_n \end{pmatrix}, \tag{3.5}$$

$M_n(\alpha,z)$, $P_n(\alpha,z)$, $R_n(\alpha,z)$, $S_n(\alpha,z)$ are known functions of α and z; the integration path Γ goes in the complex plane α along the real positive semi-axis, deviating from it only to round the real poles of the integrands in accordance with the principle of limiting absorbtion. Note that we omit the dependence of a function on t or ω if it does not lead to misunderstanding.

By expressing the Bessel functions as the half-sums of the Hankel functions of the same order we obtain the relation

$$\mathbf{J}(\alpha r)=\frac{1}{2}(\mathbf{H}^{(1)}(\alpha r)+\mathbf{H}^{(2)}(\alpha r)),\quad \mathbf{H}^{(1)}(\alpha r)=\begin{pmatrix} H_1^{(1)}(\alpha r) & 0\\ 0 & H_0^{(1)}(\alpha r)\end{pmatrix},$$

and then rewrite the representation (3.3) in the form

$$\mathbf{u}(r,z)=\frac{1}{4\pi}\int\limits_{\sigma}\mathbf{i}^{-1}\mathbf{H}^{(1)}(\alpha r)K(\alpha,z)\sum_{m=1}^{M}\hat{V}_m\mathbf{Q}_m(\alpha)\alpha d\alpha, \tag{3.6}$$

with the infinite integration path σ taken instead of the semi-infinite Γ. It is chosen by the same way, i.e. in accordance with the principle of limiting absorbtion.

Due to finite thickness of the waveguide considered, the elements of matrix $K(\alpha,z)$ are meromorphic functions of α. In the complex plane α they have a finite number of real poles $\pm\zeta_k, k=1,2,...,N_r$ and an infinite set of complex ones: $\pm\zeta_k, k=N_r+1,N_r+2,....$ The poles appear in pairs, because the functions entering $K(\alpha,z)$ are either even or odd with respect to their first variable α. We assume that the poles $+\zeta_k$ are located in the α-plane above the integration path σ, while $-\zeta_k$ lie below it, and the complex poles are arranged in order of imaginary parts increasing ($|\mathrm{Im}\,\zeta_{k+1}|\geq|\mathrm{Im}\,\zeta_k|$).

Further, from the relations (3.4) we conclude that with $r\geq r_{2M}$ the integrand in (3.6) exponentially decays in the upper half-plane of the complex plane α as $O(e^{-\mathrm{Im}\,\alpha\,(r-r_{2M})})$. This allows us to close the integration path and replace the integeral by the sum of residues. As a result, we obtain the representation of the wave field in terms of normal modes:

$$\mathbf{u}(r,z)=\sum_{k=1}^{\infty}\mathbf{u}_k(r,z),\quad r\geq r_{2M}, \tag{3.7}$$

where

$$\mathbf{u}_k(r,z)=\sum_{m=1}^{M}\hat{V}_m\mathbf{u}_{mk}(r,z) \tag{3.8}$$

and

$$\mathbf{u}_{mk}(\mathbf{r},z)=\frac{i}{2}\mathbf{i}^{-1}\mathbf{H}^{(1)}(\zeta_k r)K_k\mathbf{Q}_{mk}\zeta_k,\quad K_k\mathbf{Q}_{mk}=\mathrm{res}\,K(\alpha,z)|_{\alpha=\zeta_k}\mathbf{Q}_m(\zeta_k).$$

At that, the terms corresponding to real poles ζ_k describe guided waves going to infinity with phase and group velocities $c_{p,k}=\omega/\zeta_k$ and $c_{g,k}=d\omega/d\zeta_k$, while complex ζ_k yield inhomogeneous, exponentially decaying waves.

The substitution of (3.7) into (3.2) and use of (3.8) lead to the sought for representation of view (2.6) with

$$\mathbf{v}_k(r,z,t) = \frac{1}{\pi}\mathrm{Re}\sum_{m=1}^{M}\int_0^\infty \mathbf{u}_{mk}(r,z,\omega)\hat{V}_m(\omega)e^{-i\omega t}d\omega, \tag{3.9}$$

where

$$\hat{V}_m(\omega) = A_m\int_{t_m}^\infty f_n(t-t_m)e^{i\omega t}dt = e_m(\omega)\hat{f}_n(\omega), \quad e_m(\omega) = A_m e^{i\omega t_m} \tag{3.10}$$

are the frequency spectras of the driving parameters $V_m(t)$. Due to the assumptions imposed on the functions $\hat{f}_n(\omega)$, each $\mathbf{v}_k(r,z,t)$ approximately represents a wavepacket propagating from the source to infinity with the velocity $c_{g,k}(\omega_c)$.

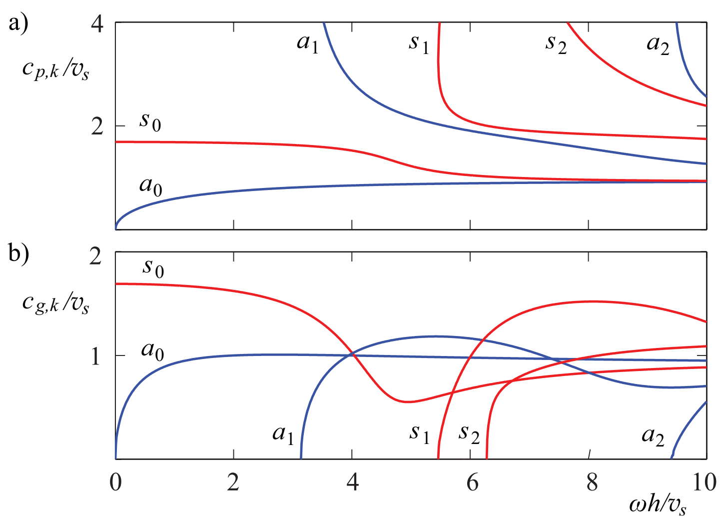

Figure 2. The phase (a) and group (b) velocities of the normal modes of the plate as functions of frequency.

As it is well known, the phase and group velocities of the normal modes of an elastic plate strongly depend of frequency, and that leads to the dispersive wave propagation. An example of the dispersion curves of an elastic plate is given in the figure 2. All the magnitudes are expressed in the dimesionless form, so the choice of the material are determined only by the Poisson's ratio $\nu = 0.3$.

4. Energy relations

In the course of time the total amount of energy $\mathcal{E}$ contained in a volume of an elastic media V is changing, i.e. $\mathcal{E}$ is a function of time. In the absence of internal sources the change is associated with the flux of energy through the surface S of the volume V. They are connected to each other by the fundamental equation

$$\frac{d\mathcal{E}}{dt} = \int_S e_n dS, \tag{4.1}$$

where $e_n = -\left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\sigma}_n\right)$ is the density of the flux through the fixed point of the surface S in the direction of the external normal n to S in this point; we use the notation $(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{n} a_k b_k^*$ for the dot product in the Hilbert space l_2^n of the column vectors of the dimension n.

Within the assumption of steady state vibrations the main energy characteristic is the time-averaged over the period of oscillations $T = 2\pi/\omega$ change of the total amount of energy (time averaged power):

$$E^h = \frac{1}{T}\int_0^T \frac{d\mathcal{E}}{dt} dt. \tag{4.2}$$

Since in the case of harmonic vibrations

$$\mathbf{v} = \mathrm{Re}\,\mathbf{u}\,\cos\omega t + \mathrm{Im}\,\mathbf{u}\,\sin\omega t,$$

$$\boldsymbol{\sigma}_n = \mathrm{Re}\,\boldsymbol{\tau}_n\,\cos\omega t + \mathrm{Im}\,\boldsymbol{\tau}_n\,\sin\omega t$$

and

$$\frac{\partial \mathbf{v}}{\partial t} = \omega(-\mathrm{Re}\,\mathbf{u}\,\sin\omega t + \mathrm{Im}\,\mathbf{u}\,\cos\omega t),$$

we rewrite (4.2) as

$$E^h = -\frac{1}{T}\int_0^T \int_S \left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\sigma}_n\right) dS dt = \frac{\omega}{2}\int_S \left((\mathrm{Re}\,\mathbf{u}, \mathrm{Im}\,\boldsymbol{\tau}) - (\mathrm{Im}\,\mathbf{u}, \mathrm{Re}\,\boldsymbol{\tau})\right) dS,$$

and finally obtain the representation

$$E^h = \frac{\omega}{2}\mathrm{Im}\int_S (\boldsymbol{\tau}_n, \mathbf{u}) dS. \tag{4.3}$$

In particular, the source energy E_0^h coming from the actuators into the substrate can be obtained by the integration over the surface $z = 0$:

$$E_0^h = \frac{\omega}{2}\mathrm{Im}\int_0^\infty \left(\mathbf{u}(r, 0), \sum_{m=1}^{M} \hat{V}_m \mathbf{q}_m(r)\right) r dr. \tag{4.4}$$

The substitution of (3.6) instead of $\mathbf{u}$ and the change of the integration order lead to the alternative form

$$E_0^h = \frac{\omega}{8\pi^2}\mathrm{Im}\int_\Gamma \left(K(\alpha, 0)\sum_{m=1}^{M} \hat{V}_m \mathbf{Q}_m(\alpha), \sum_{m=1}^{M} \hat{V}_m \mathbf{Q}_m(\alpha^*)\right) \alpha d\alpha, \tag{4.5}$$

expressing the time averaged power in terms of the Fourier-Bessel symbols. Further, since only the residues from the real poles contribute to the imaginary part of the integral (4.5), we obtain the time averaged power as the sum of the time averaged energies E_k^h carried out to infinity by the normal modes associated with the wave numbers ζ_k:

$$E_0^h = \sum_{k=1}^{N_1} E_k^h, \tag{4.6}$$

$$E_k^h = \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij}^k \, V_i V_j^* = (A^k \, \hat{\mathbf{V}}, \hat{\mathbf{V}}), \tag{4.7}$$

at that

$$A^k = (a_{ij}^k), \quad a_{ij}^k = \frac{\omega}{8\pi} \, (\mathrm{res}\, K(\alpha, 0)|_{\alpha=\zeta_k} \mathbf{Q}_j(\zeta_k), \mathbf{Q}_i(\zeta_k)) \, \zeta_k \tag{4.8}$$

is a positively defined Hermitian matrix.

In the general non-stationary case we introduce another characteristic, namely, the change of the total amount of energy during the whole actuation process:

$$E = \int_0^\infty \frac{d\mathcal{E}}{dt} dt = -\int_0^\infty \int_S \left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\sigma}_n \right) dS dt.$$

The change of the integration order, the substitution of the representation (3.1) and the similar one for $\boldsymbol{\sigma}_n$, and use of Parseval's equality give

$$E = -\int_S \int_0^\infty \left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\sigma}_n \right) dt \, dS = -\frac{1}{2\pi} \int_S \int_{-\infty}^\infty (-i\omega) \, (\mathbf{u}, \boldsymbol{\tau}_n) \, d\omega \, dS.$$

By changing the integration order back, splitting up the resulting integral over the whole real axis into the sum of interals over the half-axes and using the relations

$$\mathbf{u}(-\omega) = \mathbf{u}^*(\omega), \quad \boldsymbol{\tau}_n(-\omega) = \boldsymbol{\tau}_n^*(\omega)$$

we rewrite it as

$$E = -\frac{1}{2\pi} \int_0^\infty (-i\omega) \int_S ((\mathbf{u}, \boldsymbol{\tau}_n) - (\boldsymbol{\tau}_n, \mathbf{u})) \, dS \, d\omega = \frac{2}{\pi} \int_0^\infty \frac{\omega}{2} \, \mathrm{Im} \int_S (\boldsymbol{\tau}_n, \mathbf{u}) dS \, d\omega.$$

Finally, the comparison of the result with (4.3) allows us to rewrite it as

$$E = \frac{2}{\pi} \int_0^\infty E^h(\omega) d\omega, \quad E^h(\omega) = \frac{\omega}{2} \mathrm{Im} \int_S (\boldsymbol{\tau}_n, \mathbf{u}) dS. \tag{4.9}$$

Specifically, the total amount of energy carried by the k-th mode to infinity during the whole actuation process is expressed as

$$E_k = \frac{2}{\pi} \int_0^\infty E_k^h(\omega) d\omega = \frac{2}{\pi} \sum_{i=1}^{M} \sum_{j=1}^{M} \int_0^\infty a_{ij}^k(\omega) \, \hat{V}_i(\omega) \hat{V}_j^*(\omega) d\omega, \tag{4.10}$$

or, alternatively, using the the matrix notation, as

$$E_k = \frac{2}{\pi} \int_0^\infty \left(A^k(\omega) \hat{\mathbf{V}}(\omega), \hat{\mathbf{V}}(\omega) \right) d\omega. \tag{4.11}$$

This is the sought for representation of the total amount of energy carried out to infinity by the k-th mode during the actuation process, entering the objective function (2.7).

5. Selective radiation

Let us examine the structure of representation (4.11) in the particular case of driving parameters $V_m(t)$ taken as specified by (2.9). The substitution of the representations (3.10) into (4.11) implies

$$E_k = \frac{2}{\pi}\int\limits_0^\infty \left(A^k(\omega)\mathbf{e}(\omega), \mathbf{e}(\omega)\right) |\hat{f}_n(\omega)|^2 d\omega, \tag{5.1}$$

where $\mathbf{e}(\omega) = (e_1(\omega), e_2(\omega), \ldots, e_M(\omega))^T$. As we assumed in the problem statement section, the functions $\hat{f}(\omega)$ form a delta like sequence, i.e. with a large enough n the value of the integral is determined only by a small neighbourhood of the point ω_c. Hence, an approximation for $\mathbf{e} = \mathbf{e}(\omega_c)$ is obtained as the solution to the problem

$$\nu_0(\mathbf{e}) = \frac{(A\mathbf{e}, \mathbf{e})}{(B\mathbf{e}, \mathbf{e})} \to \max, \quad A = \sum_{k\in H} A^k(\omega_c), \quad B = \sum_{k\in N\setminus H} A^k(\omega_c) \tag{5.2}$$

with $\mathbf{e} = (e_1, e_2, \ldots, e_M)^T$ being a constant column vector. And this is nothing else then the problem of maximization of the contrast of radiation of the selected modes at the given angular frequency $\omega = \omega_c$ in the time harmonic case.

The extremal points of $\nu_0(\mathbf{e})$ satisfy the extremum conditions

$$\frac{\partial \nu_0(\mathbf{e})}{\partial e_j} = 0, \quad j = 1, 2, \ldots, M \tag{5.3}$$

which lead to the nonlinear system

$$(A - \nu_0 B)\mathbf{e} = 0 \tag{5.4}$$

with respect to the vector of unknown parameters $\mathbf{e}$. If matrix B is non-singular ($\det B \neq 0$), the latter expression is rewritten as

$$R\mathbf{e} - \nu_0\mathbf{e} = 0, \quad R = B^{-1}A, \tag{5.5}$$

and the problem is reduced to searching for the eigenvector $\mathbf{e}_s$ of matrix R corresponding to its maximal eigenvalue λ_s: $R\mathbf{e}_s = \lambda_s\mathbf{e}_s$. In such a case $A\mathbf{e}_s = \lambda_s B\mathbf{e}_s$, hence $\nu_0(\mathbf{e}_s) = \lambda_s$ and the equation (5.5) holds identically.

If matrix B is singular ($\det B = 0$), then its eigenvector $\mathbf{e}_0$ associated with its eigenvalue $\lambda_0 = 0$ being taken as $\mathbf{e}$ provides the full damping of undesirable radiation at the given frequency: $(B\mathbf{e}_0, \mathbf{e}_0) = 0$. In the event of k_0 eigenvectors $\mathbf{e}_k$ associated with $\lambda_0 = 0$, the vector $\mathbf{e}_s$ maximizing the form $(A\mathbf{e}, \mathbf{e})$ is to be searched as a linear combination of those vectors:

$$\mathbf{e}_s = \sum_{k=1}^{k_0} c_k\mathbf{e}_k.$$

By imposing an additional restriction $||\mathbf{e}_s|| = 1$, we reduce this problem to the one similar to (5.3), but with the matrices

$$A^0 = (a^0_{ij}), \quad a^0_{ij} = (A\mathbf{e}_j, \mathbf{e}_i), \qquad B^0 = (b^0_{ij}), \quad b^0_{ij} = (\mathbf{e}_j, \mathbf{e}_i)$$

and the vector column $\mathbf{c} = (c_1, c_2, \ldots, c_{k_0})$ being taken instead of A, B and $\mathbf{e}$. At that, B_0 is non-singular as the gramian of a linearly-independent system of vectors.

To derive sufficient conditions for making matrix B singular, i.e. the conditions of secure suppression of undesirable modes at the given frequency, let us analyse its structure with more details. By virtue of the representation (4.8) its elements have the view

$$b_{ij} = \sum_{k \in N \setminus H} u_{ki} u_{kj}^*, \quad u_{ki} = (-im_k)^{\frac{1}{2}} Q_{ri}(\zeta_k) + (-r_k)^{\frac{1}{2}} Q_{zi}(\zeta_k),$$

where $m_k = \operatorname{res} M(\alpha, 0)|_{\alpha=\zeta_k}$ and $r_k = \operatorname{res} R(\alpha, 0)|_{\alpha=\zeta_k}$. It means that the matrix B is nothing but the gramian of M vectors of size $L = \operatorname{Card} N \setminus H$. If $M > L$, the system of this vectors is linearly dependent, hence $\det B \equiv 0$. If $M < L$, then, in general, $\det B \neq 0$. In other words, M sources are enough to suppress completely $L < M$ traveling modes at the given frequency.

As soon as the vector $\mathbf{e}$ satisfying the maximization problem (5.2) has been calculated, we use the second of the relations (3.10) to obtain an approximate solution to the initial general maximization problem:

$$A_m = |e_m|, \quad t_m = \omega_c^{-1} \arg e_m \quad \text{or} \quad A_m = -|e_m|, \quad t_m = \omega_c^{-1}(\pi + \arg e_m),$$

depending on whether $0 \leq \arg e_m \leq \pi$ or $-\pi \leq \arg e_m \leq 0$. At that, the greater is the number n, the more accurate is the approximation.

Within the boundaries of the simplified model for the actuator described in the statement section the contact stresses take the form (2.4), that leads to the representations

$$Q_{rm}(\alpha) = 2\pi i \int_0^\infty q_m(r) J_1(\alpha r) r dr = i\tau_0 \left[J_1(\alpha r_{2m}) - J_1(\alpha r_{1m}) \right], \quad Q_{zm}(\alpha) = 0$$

for the Fourier-Bessel symbols of the components of the contact stresses. This means that the positively defined hermitian matrices A, B and A_0, B_0 have pure real components, and, as consequence, their eigenvectors are also pure real, correct to a complex factor. Therefore, we always have zero time shifts t_m, and selective radiation is achieved only by the proper choice of the amplitudes A_m of the driving signals.

6. Numerical examples

Let us consider a steel plate ($Y = 210$ GPa, $\nu = 0.3$, $\rho = 7800$ kg m^{-3}) of thickness $h = 5$ mm with five ring-shaped piezoelectric transducers bonded to its surface. The transdusers have equal widths and are equally spaced ($r_{1m}/h = 3m - 2$ and $r_{2m}/h = 3m$). They are excited using n-cylce sinusoidal tonebursts of the form (2.11) with correspondingly chosen amplitudes A_m and time delays t_m.

The dispersion curves of the plate are shown in figure 2. As it was stated in the problem statement section, we use a simplified model for the actuators, so only the low

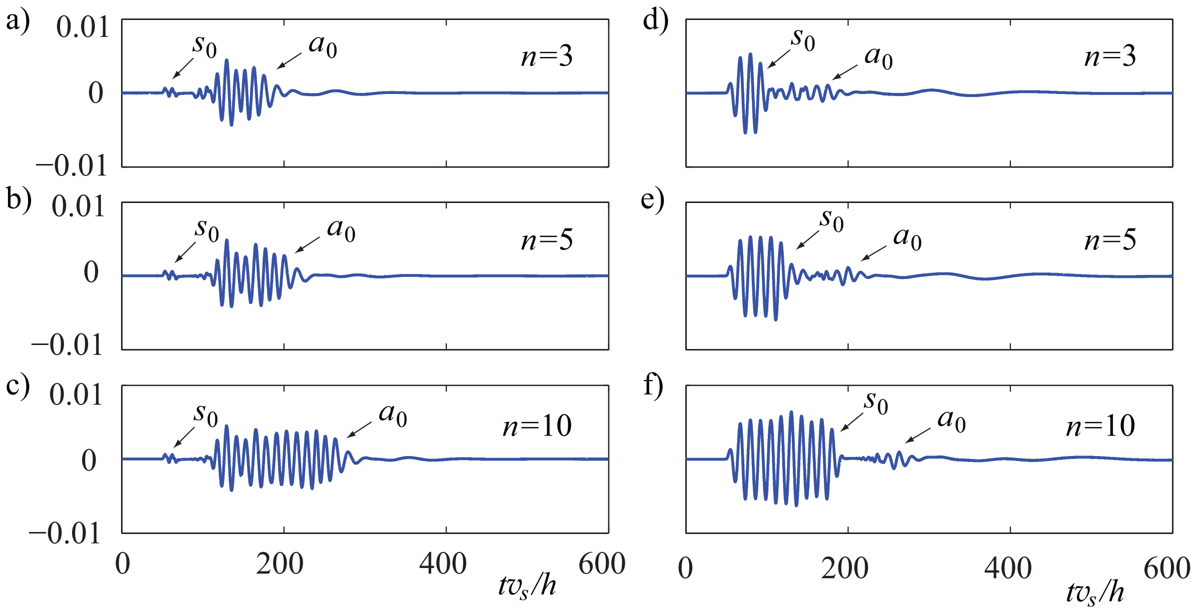

Figure 3. The radial displacements u_r/h of the point $r = 100h$ at the top surface of the plate as functions of time. Selective radiation of the a_0 mode (a-c) or the s_0 mode (d-f) was required.

	selective a_0 radiation			selective s_0 radiation		
n	3	5	10	3	5	10
$E_1/(\rho h^3 v_s^2)$	0.1444	0.1996	0.3389	0.0108	0.0107	0.0107
$E_2/(\rho h^3 v_s^2)$	0.0014	0.0014	0.0014	0.0597	0.1076	0.2274
E_1/E	0.9904	0.9930	0.9959	0.1532	0.0904	0.0449
E_2/E	0.0096	0.0070	0.0041	0.8468	0.9096	0.9551

Table 1. The distributions of energy among the a_0 and s_0 modes for various conditions of radiation.

	A_1	A_2	A_3	A_4	A_5
selective a_0 radiation	0.5291	−0.2787	0.3966	−0.2861	0.6350
selective s_0 radiation	0.4406	0.2639	−0.3645	−0.4629	−0.6238

Table 2. The cooresponding amplitudes of the driving signals.

frequency range ($0 \leq \omega h/v_s \leq \pi$) is of interest. Only two modes are propagating here, namely, the first antisymmetric mode a_0 and the first symmetric mode s_0. Our goal is to achieve the radiation of the only desirable mode using the driving signals with a given central frequency, for all the examples in this section it was chosen as $\omega_c h/v_s = 0.5$.

As an illustration of the proposed approach, figure 3 shows the radial displacements u_r/h of the point $r = 100h$ at the top surface of the plate as functions of time. For the subplots (a-c), we aimed to radiate only the first antisymmetric mode a_0 using sinusoidal tonebursts with different number of cycles. Similarly, the subplots (d-f) were obtained from the condition of the selective radiation of only the first symmetric mode s_0. The vector $\mathbf{e}$ is always normalized so that $|\tau_0\mathbf{e}| = 1$. In both cases, one can clearly see the main wavepacket corresponding to the desirable mode.

The more precise information on the distribution of energy among the modes is given in the table 2. It is interesting to note that with the increasing number of cycles the total amount of energy carried by the selected mode noticeably increase, whereas for the undesirable one it remains almost the same, and the increase is caused by the growth of the length of the wavepacket.

7. Concluding remarks

An omni-derectional multi-element transducer composed of a system of coaxial ring-shaped piezoelectric elements actuated by n-cycled sinusoidal tonebursts has been presented. It is capable of generating a single required Lamb wave mode in an elastic plate and suppressing undesirable ones. Mode selection is achieved by a special choice of the time delays and the amplitudes of the driving signals, which are effectively calculated using the information on the geometrical configuration of the system, the physical properties of the plate and the central frequency of the excitation tonebursts.

The method for the determination of these parameters is based on strict analytical considerations. In the limiting case of time-harmonic problem it allows to generate only a single required mode, and to completely eliminate undesirable ones, provided that a number of the rings is greater than the number of the modes to be suppressed. It is shown that within the range of aplicability of a simplest model for the piezoelectric elements, i.e. when the actuating force can be replaced by a system of concentrated forces applied at the locations the edges of the element, no time delays are needed, and a single required mode can be always excited only by an approprite choice of the amplitudes of the input signals.

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